

# Some Properties of Strongly Regular Graphs

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## Abstract

An approach to the enumeration of feasible parameters for strongly regular graphs is described, based on the pair of structural parameters  $(a, c)$  and the positive eigenvalue  $e$ . The Krein bound ensures that there are only finitely many possibilities for  $c$ , given  $a$  and  $e$ , and the standard divisibility conditions can be used to reduce the possibilities further. Many sets of feasible parameters appear to be accidents of arithmetic, but in some cases the conditions are satisfied for algebraic reasons. As an example, we discuss an infinite family of feasible parameters for which the corresponding graphs necessarily have a closed neighbourhood as a star complement for  $e$ .

## 1. Introduction

A *strongly regular* graph  $X$  is characterized by three parameters  $k$ ,  $a$ , and  $c$ , according to the rules

- $X$  is regular with degree  $k$ ;
- any two adjacent vertices have  $a$  common neighbours;
- any two non-adjacent vertices have  $c$  common neighbours.

We shall discuss only cases where

$$k \geq 3, \quad k > c \geq 1.$$

These conditions rule out some simple examples, such as the complete bipartite graphs  $K_{k,k}$ , which have  $c = k$ . Thus, when we say that  $X$  is an  $\text{SR}(n, k, a, c)$  graph we mean that it is a non-bipartite connected graph with  $n$  vertices, and the parameters  $k, a, c$  satisfy the conditions listed above.

Since about 1970 many lists of feasible parameters have been constructed. Brouwer's list in the *Handbook of Combinatorics* contains all possibilities with  $n \leq 280$ , and his online version currently extends to  $n \leq 1300$  [2]. These lists contain some parameters that are impossible according to the so-called Krein bounds, and the method to be developed here will start by excluding these cases.

We shall need some standard notation and theory [9]. If  $v$  is a vertex of an  $\text{SR}(n, k, a, c)$  graph  $X$  we denote the subgraphs induced by the sets of vertices at distances 1 and 2 from  $v$  by  $X_1(v)$  and  $X_2(v)$  respectively. The sizes of  $X_1(v)$ ,  $X_2(v)$ , and  $X$  are given by

$$|X_1(v)| = k, \quad |X_2(v)| = \ell = \frac{k(k-a-1)}{c}, \quad |X| = n = 1 + k + \ell.$$

The complement  $X^*$  of a strongly regular graph  $X$  is strongly regular with  $k^* = \ell$  and  $\ell^* = k$ . We shall normally deal with a representative of the complementary pair that satisfies  $k \leq \ell$ .

The adjacency matrix  $A$  of  $X$  satisfies the equations

$$AJ = kJ \quad \text{and} \quad A^2 - (a-c)A - (k-c)I = cJ,$$

where  $I$  is the identity matrix and  $J$  is the all-1 matrix. It follows that the eigenvalues of  $A$  are  $k$  (with multiplicity 1) and the roots  $\lambda_1, \lambda_2$  of the equation  $\lambda^2 - (a-c)\lambda - (k-c) = 0$ . We shall not concern ourselves with situation when the discriminant of this equation is irrational, because the possibilities in that case are very limited [9]. So we shall assume that there

is a positive integer  $s$  such that  $s^2 = (a - c)^2 + 4(k - c)$ , and the eigenvalues are the integers

$$k = \frac{s^2 - (a - c)^2}{4} + c \quad \lambda_1 = \frac{a - c + s}{2}, \quad \lambda_2 = \frac{a - c - s}{2}.$$

Elementary arguments show that the multiplicities  $m_1, m_2$  of  $\lambda_1, \lambda_2$  are given by the formulae

$$m_1 = \frac{k}{2cs} \left( (k + c - a - 1)(s + c - a) - 2c \right), \quad m_2 = \frac{k}{2cs} \left( (k + c - a - 1)(s - c + a) + 2c \right).$$

Given that the graph  $X$  exists, these formulae must represent integers. If we take the basic parameters to be  $a, c$ , and  $e = \lambda_1$ , then  $k = (e + 1)c + e(e - a)$  and  $s = c + 2e - a$ , so the conditions can be formulated in terms of those parameters. Further relationships linking the structural parameters, the eigenvalues, and the multiplicities are well known, and can be found in the standard texts [7], [9 p.244]. A formulation suitable for our purposes will be presented in Section 3. But first we shall look at another condition, which effectively bounds  $c$  in terms of  $a$  and  $e$ .

## 2. The Krein bounds and their consequences

Conditions asserting the non-negativity of the so-called *Krein parameters* arise in the general theory of distance-regular graphs [4]. In the strongly regular case we have two parameters  $K_1$  and  $K_2$  which, after some elementary algebra, can be written in terms of  $k, \lambda_1, \lambda_2$  [9]:

$$K_1 = (k + \lambda_1)(\lambda_2 + 1)^2 - (\lambda_1 + 1)(k + \lambda_1 + 2\lambda_1\lambda_2).$$

$$K_2 = (k + \lambda_2)(\lambda_1 + 1)^2 - (\lambda_2 + 1)(k + \lambda_2 + 2\lambda_1\lambda_2).$$

Here we shall normally consider the member of a complementary pair for which the non-negativity of  $K_2$  is the effective condition.

The formulae given in the previous section imply that when  $\lambda_1 = e$  we have  $k = e(e - a) + (e + 1)c$  and  $\lambda_2 = a - c - e$ . Thus we can express  $K_2$  in terms of  $a, c$ , and  $e$ ; in fact it is a quadratic in  $c$ :

$$K_2(c) = P + Qc - ec^2$$

where  $P = (e + 1)(e - a)(e^2 - e + a)$  and  $Q = e^3 + (2a + 1)e + a$ .

**Theorem 2.1** The Krein condition for  $K_2$  implies that

$$c \leq \begin{cases} e^2 + e + 2a & \text{if } e \geq 3; \\ e^2 + e + 3a & \text{if } e = 1, 2. \end{cases}$$

In particular, when  $a = 0$  we have  $c \leq e(e + 1)$ .

*Proof* The Krein condition states that  $K_2 \geq 0$ . Since  $K_2$  is a quadratic function of  $c$  with a maximum at  $c = Q/2e > 0$  it follows that  $c$  must not exceed the larger root  $c_0$  of the equation  $K_2(c) = 0$ .

We can estimate  $c_0$  by evaluating  $K_2$ . When  $c = e(e + 1)$  we have

$$k = e(e^2 + 3e - a + 1) \quad \text{and} \quad K_2 = a(e + 1)(e^2 + 3e - a).$$

Since we assume that  $k \geq 3$ , we have  $K_2(e^2 + e) \geq 0$ . There is equality when  $a = 0$  so  $c_0 = e(e + 1)$  in that case. Similarly, we find

$$K_2(e^2 + e + a) = 2ae(e + 2),$$

$$K_2(e^2 + e + 2a) = a(e(5 - e^2) + a(1 - e)).$$

When  $a \geq 1$  the first value is positive and the second is negative if  $e \geq 3$ , so  $c_0$  lies strictly between  $e^2 + e + a$  and  $e^2 + e + 2a$ .

When  $e = 1$  and  $e = 2$  the values of  $K_2(e^2 + e + 3a)$  are  $2a(1 - a)$  and  $-6a(a + 2)$  respectively, from which the result follows.  $\square$

**Example 2.2** In the case  $a = 0$  we have  $c \leq e^2 + e$ , and the graph is an SRNT (strongly regular, no triangles) graph. This case was studied in earlier papers [1], using an approach based on the parameters  $(a, c, e)$ .

**Example 2.3** In the case  $a = e$  we have  $P = 0$ , and the equation  $K_2(c) = 0$  has roots 0 and

$$c_0 = Q/e = e^2 + 2e + 2.$$

In particular, when  $e = a = 1$  we have  $c \leq 5$ , and in fact graphs exist only when  $c = 2, 3, 5$ . The corresponding values of  $n$  are 9, 15, 27 and the graphs are  $LK_{3,3}$ ,  $LK_6$ , and the co-Schläfli graph [9]. Note that the parameter-set  $n = 63, k = 22, a = 1, c = 11$  also occurs in Brouwer's list [2], only to be ruled out by the Krein bound.  $\square$

### 3. The integrality conditions

We now formulate the ‘standard’ integrality conditions. As in Section 2, we take the basic parameters to be  $a, c$  and  $e$ .

**Theorem 3.1** Suppose  $X$  is an  $\text{SR}(n, k, a, c)$  graph with  $\lambda_1 = e$ , and let

$$D = e(e+1)(e-a)(e-a-1), \quad F = (e+1)(e^2+2e-a)(e^2+3e-a).$$

Then  $D$  is a multiple of  $c$  and  $F$  is a multiple of  $c+2e-a$ .

*Proof* If  $X$  exists then the ‘multiplicities’  $m_1$  and  $m_2$  must be integers.

Since  $m_1 + m_2 = k + \ell = n - 1$ , the condition that  $m_1 + m_2$  is an integer is equivalent to the integrality of  $\ell = |X_2(v)|$ . We have

$$\ell = \frac{k}{c}(k-a-1),$$

and substituting  $k = (e+1)c + e(e-a)$  leads to the formula

$$\ell = (e+1)^2c + (e+1)(2e^2 - 2ae - a - 1) + \frac{D}{c},$$

where  $D$  is the expression given above. Thus  $D$  is a multiple of  $c$ .

The formula for  $m_2$  reduces to

$$m_2 = \frac{k((k+c-a-1)e+c)}{c(c+2e-a)} = \frac{k(k-e)(e+1)}{c(c+2e-a)}.$$

Substituting for  $k$ , we find (after some elementary algebra)

$$m_2 = (e+1)^3 - \frac{Ec - De}{c(c+2e-a)},$$

where  $E = (e+1)^2(ae+3e-a)$ .

Given that  $D$  is a multiple of  $c$ , say  $D = cc'$ , the second term is an integer if  $E - ec'$  is a multiple of  $c+2e-a$ . If  $\theta$  is the integer such that  $E - ec' = \theta(c+2e-a)$ , then

$$\begin{aligned} (E - \theta c)(c+2e-a) &= E(2e-a) + De \\ &= (e+1)(e^2+2e-a)(e^2+3e-a) = F, \end{aligned}$$

so  $F$  is a multiple of  $c+2e-a$ , as claimed.  $\square$

In certain cases the conditions can be simplified.

**Corollary 3.2** If  $e = a$  the sole condition is that  $c+e$  divides  $e^2(e+1)^2(e+2)$ , and if  $e = a+1$  the sole condition is that  $c+e+1$  divides  $(e+1)^3(e^2+e+1)$ .

*Proof* When  $e = a$  or  $e = a + 1$  we have  $D = 0$ , so the first condition is trivially satisfied. Since  $c' = 0$ , the sole condition is that  $F$  is a multiple of  $c + 2e - a$ , which reduces to the forms stated.  $\square$

#### 4. Enumeration of feasible parameters

The foregoing results enable feasible parameters to be calculated systematically. The method is to fix  $a$  and  $e$ , and find those  $c$  which lie in the range specified in Theorem 2.1 and satisfy the conditions

$$c \mid D \quad \text{and} \quad c + 2e - a \mid F.$$

**Example 4.1** When  $a = 1$  and  $e = 4$  we require that  $1 \leq c \leq 22$ ,  $c$  divides  $120 = 2.3.4.5$ , and  $c + 7$  divides  $3105 = 3^3.5.23$ . It is easy to check that the only possibilities are  $c = 2, 8, 20$ . The corresponding values of  $(n, k)$  are  $(243, 22)$ ,  $(378, 52)$ , and  $(729, 112)$ . Here we have an atypical situation, because graphs are known to exist for all three sets of feasible parameters [2].  $\square$

Numerical evidence suggests that, given  $a$  and  $e$ , the number of  $c$  satisfying the required conditions is (at worst) a linear function of  $e$ . However, the situation is obscured by the notoriously complicated multiplicative structure of the integers, which means that many feasible values of  $c$  are accidents of arithmetic. More order can be imposed by regarding  $e$  as an indeterminate variable and working in the ring  $\mathbb{Z}[e]$ . If we fix  $a \in \mathbb{Z}$  and regard the expressions

$$D_a(e) = e(e+1)(e-a)(e-a-1), \quad F_a(e) = (e+1)(e^2+2e-a)(e^2+3e-a),$$

as elements of  $\mathbb{Z}[e]$ , then we can look for divisors  $c$  of  $D_a(e)$  in  $\mathbb{Z}[e]$  such that  $c + (2e - a)$  is a divisor of  $F_a(e)$ . In the light of Theorem 2.1 we may also impose the condition that the divisors have degree at most 2. When  $a = 0$  there are several possibilities, but for any positive  $a$  there is only one possibility,

$$c = e(e+1), \quad c + (2e - a) = e^2 + 3e - a.$$

We shall refer to this as the *algebraically feasible* case, and discuss it in more detail in Section 6.

The foregoing results also facilitate compilation of lists of strongly regular graphs in order of the number of vertices,  $n$ . Since  $n = 1 + k + \ell$ , it follows from the formula for  $\ell$  obtained in the proof of Theorem 3.1 that we can write  $n$  in the form

$$n = Gc + H + \frac{D}{c},$$

where  $D$  is as before and

$$G = (e + 1)(e + 2),$$

$$H = 2e^3 + (3 - 2a)e^2 - (1 + 4a)e - a.$$

For given  $a$  and  $e$ , consider  $n$  as a function of  $c$  in the range stated in Theorem 2.1. Clearly  $n$  has only one turning point, a minimum, at the point where  $c^2 = D/G$  and  $n$  takes the value  $H + 2\sqrt{DG}$ . Roughly speaking, the minimum occurs near the point  $c = e$ , and the minimum value  $n_{min}$  is about  $4e^3$ .

The general behaviour of  $n$ , and in particular the location of the maximum  $n_{max}$ , can be inferred from a few calculations. With a few exceptions, the largest feasible value of  $c$  occurs in the algebraically feasible case  $c = e(e + 1)$ , in which case  $n = (e^2 + 3e - a)^2$ .

These bounds provide an effective method of tabulating the results. For  $a = 0$  (the SRNT case) and  $1 \leq e \leq 10$  the bounds are as follows:

$e$	1	2	3	4	5	6	7	8	9	10
$n_{min}$	4	50	154	342	638	1066	1650	2413	3381	4577 .
$n_{max}$	16	100	324	784	1600	2916	4900	7744	11664	16900

Suppose we wish to list all the feasible parameters for SRNT graphs with at most 1000 vertices. According to the table, we need only carry out the calculation for  $1 \leq e \leq 5$ , since  $n_{min}(6)$  is greater than 1000. Similarly, if we list the feasible parameters for  $1 \leq e \leq 10$ , the list will contain all possibilities with fewer than 6025 vertices, since  $n_{min}(11) = 6025$ .

## 5. Star complements

The numerical conditions derived above are necessary, but not sufficient, for the existence of a strongly regular graph. For small values of the parameters they appear to be a remarkably good guide, because graphs can be constructed for a high proportion of the feasible sets of parameters. Whether this is true in general remains a mystery.

One approach to the questions of existence and uniqueness of an  $SR(n, k, a, c)$  graph is based on the reconstruction of the graph from a hypothetical subgraph. In this section we shall discuss the possibility of using a *star complement* as the subgraph. This theory has been developed extensively by Cvetković, Rowlinson and Simić [8]. We shall also employ arguments similar

to those used in the method of subconstituents, as described by Godsil and Royle [9].

Let  $P \cup Q$  be a partition of the vertex-set of a graph  $X$ . The adjacency matrix of  $X$  is partitioned correspondingly in the form

$$\begin{pmatrix} A_P & B^T \\ B & A_Q \end{pmatrix},$$

where  $A_P$ ,  $A_Q$ , are the adjacency matrices of the subgraphs induced by  $P$  and  $Q$ , and  $B$  specifies the edges with one vertex in  $P$  and one vertex in  $Q$ . In general there is no relationship between these submatrices, but when the partition is chosen in a particular way, there is.

Let  $X$  be a graph with  $n$  vertices that has an eigenvalue  $e$  with multiplicity  $m$ . It is easy to show that removing a vertex from  $X$  reduces the multiplicity of  $e$  by 1, at most. Hence the maximum cardinality of an induced subgraph that does not have  $e$  as an eigenvalue is  $n - m$ . Such a subgraph is said to be a *star complement* for  $e$  in  $X$ . It can be shown that star complements always exist. For our purposes the following result [8] is fundamental.

**Theorem 5.1 (The Reconstruction Theorem)** If  $Q$  is a star complement for  $e$  in  $X$ , then the submatrices  $A_P$ ,  $A_Q$  and  $B$  satisfy the equation

$$eI - A_P = B^T (eI - A_Q)^{-1} B.$$

(Here we adopt the convention, strictly incorrect, of using the same notation for a subset  $Q$  of the vertices of  $X$  and the subgraph induced by  $Q$ .  $\square$ )

In a nutshell, this equation says that the edges of  $X$  having both vertices in  $P$  can be reconstructed, given  $e$  and the edges with at least one vertex in  $Q$ .

Suppose we are trying to construct a strongly regular graph  $X$  with parameters  $(a, c, e)$ . The degree and cardinality of  $X$ , and the cardinality of a star complement for  $e$ , are thus

$$k = (e + 1)c + e(e - a), \quad n = Gc + H + c',$$

$$n - m = m_2 + 1 = \frac{k(k - e)(e + 1)}{c(c + 2e - a)} + 1,$$

where  $cc' = D$  and  $D, G, H$  are given by the formulae in Sections 3 and 4. If we can identify a set  $Q$  of  $n - m$  vertices that induces a subgraph which does not have  $e$  as an eigenvalue, then the construction of  $X$  depends on finding a suitable matrix  $B$ . This defines the edges with exactly one vertex in  $P$  and,



by the Reconstruction Theorem, the edges with both vertices in  $P$ . If there is a unique  $B$  compatible with the given parameters, then  $X$  is unique. If there is no such  $B$ , then  $X$  cannot exist.

The fact that the reconstructed graph  $X$  is strongly regular places severe restrictions on  $B$ .

**Lemma 5.2** Let  $X$  be a strongly regular graph with parameters  $(a, c, e)$  and suppose  $Q$  is a star complement for  $e$ , with adjacency matrix  $A_Q$ . Then the matrix  $B$  is such that  $BB^T = R$ , where

$$R = cJ + e(e + c - a)I + (a - c)A_Q - A_Q^2.$$

*Proof* The adjacency matrix

$$A = \begin{pmatrix} A_P & B^T \\ B & A_Q \end{pmatrix},$$

satisfies the quadratic equation given in Section 2. Taking the submatrices in the second row and column, and substituting  $k = (e + 1)c + e(e - a)$ , gives the result.  $\square$

The matrix  $R$  depends only on  $A_Q$ , which we assume known. The problem is to find a suitable matrix  $B$  such that  $BB^T = R$ , which is only possible when  $R$  is positive semi-definite. Variants of this problem are studied in other branches of mathematics, but our problem has some special features. Specifically,  $B$  must be a  $(0, 1)$  matrix with  $m$  columns.

**Example 5.3** In the case  $a = 0$ ,  $c = 1$ ,  $e = 1$  we have  $k = 3$ ,  $n = 10$ , and  $n - m = 5$ . The parameters imply that  $X$  must contain an induced 5-cycle  $Q$ , for which the adjacency matrix satisfies  $A_Q^2 + A_Q - I = J$ . Since  $e = 1$  is not an eigenvalue of  $A_Q$ ,  $Q$  is a star complement for  $e$ .

In this case

$$R = J + 2I - A_Q - A_Q^2 = I.$$

Clearly, a  $5 \times 5$   $(0, 1)$ -matrix satisfying the equation  $BB^T = R$  is  $B = I$ . By the Reconstruction Theorem,  $I - A_P = (I - A_Q)^{-1}$ , and using the quadratic equation for  $A_Q$  it is easy to check that

$$(I - A_Q)^{-1} = A_Q + 2I - J, \quad \text{hence} \quad A_P = J - I - A_Q.$$

This means that  $P$  is a pentagram, and the familiar picture of the Petersen graph is obtained. A more detailed analysis leads to the conclusion that this is the only possibility, up to isomorphism.  $\square$

**Example 5.4** In the case  $a = 1, c = 3, e = 1$  we have  $k = 5, n = 15$ , and  $n - m = 6$ . Since  $K_{3,3}$  has six vertices and its eigenvalues are  $3, 0, -3$ , we can try it as a star complement  $Q$ . (In fact it can be shown that the parameters require an induced subgraph of this form, but it is enough to show that the construction works on this assumption.) We have

$$A_Q = \begin{pmatrix} O & J \\ J & O \end{pmatrix}, \quad A_Q^2 = \begin{pmatrix} 3J & O \\ O & 3J \end{pmatrix},$$

$$R = 3J + 3I - 2A_Q - A_Q^2 = \begin{pmatrix} 3I & J \\ J & 3I \end{pmatrix}.$$

A  $6 \times 9$   $(0, 1)$ -matrix  $B$  satisfying  $BB^T = R$  is

$$B = \begin{pmatrix} I & I & I \\ E_1 & E_2 & E_3 \end{pmatrix} \quad \text{where}$$

$$E_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Applying the Reconstruction Theorem, we find

$$A_P = \begin{pmatrix} O & F & F \\ F & O & F \\ F & F & O \end{pmatrix} \quad \text{where} \quad F = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

from which it is clear that  $P = LK_{3,3}$ . □

## 6. The algebraically feasible case

In some cases a suitable star complement is suggested by the structure of the graph. For example, the *closed neighbourhood*  $N = \{v\} \cup X_1(v)$ , where  $v$  is any vertex in a strongly regular graph with parameters  $(a, c, e)$ , consists of  $k + 1$  vertices,  $k$  of which have degree  $a + 1$  and one of which has degree  $k$ . The following results show that  $N$  is potentially a star complement for  $e$ , for infinitely many sets of feasible parameters.

The formula obtained in Lemma 5.2 can be reduced to a more obvious form when  $Q = N$ . Suppose  $w$  is any vertex in  $X_1(v)$ . Then

$$(A_N)_{vv} = 0, \quad (A_N)_{vw} = 1, \quad (A_N^2)_{vv} = k, \quad (A_N^2)_{vw} = a.$$

Hence

$$R_{vv} = c + e(e + c - a) - k = 0, \quad R_{vw} = c + (a - c) - a = 0.$$

Removing row  $v$  and column  $v$  from  $R$ ,  $A_N$ , and  $A_N^2$ , we get  $k \times k$  matrices  $R_b$ ,  $A_1$ , and  $A_2$  such that

$$R_b = cJ + e(e + c - a)I + (a - c)A_1 - A_2.$$

(Note that  $A_2$  is not the same as  $(A_1)^2$ .)

The matrix  $B_b$  obtained from  $B$  by removing row  $v$  (which must clearly be a row of 0s) is such that  $B_b B_b^T = R_b$ . So  $B_b$  is simply the  $(0, 1)$ -matrix of size  $k \times \ell$  that specifies the edges between  $X_1(v)$  and the putative  $X_2(v)$ . The critical fact is that if  $N$  is a star complement, the entire structure of  $X_2(v)$  can be reconstructed from  $B_b$ .

Recall that in Section 4 the parameters  $(a, c, e)$  were shown to be ‘algebraically feasible’ whenever  $c = e(e + 1)$ . In that case the corresponding values of  $k$  and  $n$  are

$$k = e(e^2 + 3e - a + 1), \quad n = (e^2 + 3e - a)^2.$$

**Theorem 6.1** Let  $X$  be a strongly regular graph with parameters  $(a, c, e)$  such that  $c = e(e + 1)$ . Then for all  $a$  and all  $e > a$  the closed neighbourhood  $N$  of a vertex is a star complement for  $e$  in  $X$ .

*Proof* We have  $m_2 = k(k - e)(e + 1)/c(c + 2e - a)$ , and elementary algebra shows that if  $c = e(e + 1)$  then  $(k - e)(e + 1) = c(c + 2e - a)$ . Hence  $m_2 = k$  and  $n - m = m_2 + 1 = k + 1$ , so  $N$  has the right size.

It remains to show that  $e$  is not an eigenvalue of  $N$ . The first subconstituent  $X_1$  is a regular graph with degree  $a$  and  $k$  vertices. Hence its characteristic polynomial has the form

$$P(x) = (x - a)^p R(x),$$

where  $p$  is the number of components, and the zeros of  $R(x)$  are such that  $|x| \leq a$ . The closed neighbourhood  $N$  is obtained by joining one new vertex to all vertices of  $X_1$ , and by a standard result its characteristic polynomial is

$$(x - a)^{p-1} R(x)(x^2 - ax - k).$$

Hence all the eigenvalues of  $N$  satisfy  $|x| \leq a$ , except possibly for the roots of  $x^2 - ax - k = 0$ . However, putting  $x = e$  we have

$$e^2 - ae - k = e^2 - ae - e(e^2 + 3e - a - 1) = -e^2(e + 3) \neq 0.$$

So  $e$  is not a root, and if  $e > a$ , then  $e$  is not an eigenvalue of  $N$ .  $\square$

**Example 6.2** When  $a = 0$ ,  $N$  is the graph  $K_{1,k}$ . In the cases  $e = 1$  and  $e = 2$  the closed neighbourhood can be used to reconstruct the graphs uniquely [8], resulting in the Clebsch and Higman-Sims graphs. However  $e = 3$  corresponds to a graph with 324 vertices, and it has been shown by other means that the graph does not exist [11, 15]. When  $a = 1$ ,  $N$  is a ‘windmill’ with  $k/2$  triangles, and in the case  $e = 2$  Stevanović and Milosević [14] used the reconstruction method to prove that there is a unique graph. This graph had previously been studied by several authors [3, 5].  $\square$

In Example 6.2 we noted two graphs with  $a = e - 1$ . When  $e = 1$  we have the Clebsch graph  $\text{SR}(16, 5, 0, 2)$ , and when  $e = 2$  we have a graph  $\text{SR}(81, 20, 1, 6)$ . It is natural to ask if these graphs are part of a family with algebraically feasible parameters of the form

$$n = (e + 1)^4, \quad k = e(e^2 + 2e + 2), \quad a = e - 1, \quad c = e(e + 1).$$

It follows from Theorem 6.1 that, for these parameters, the closed neighbourhood  $N$  is a star complement for  $e$ . The matrix  $B_b$  must satisfy  $B_b B_b^T = R_b$  where, according to Lemma 5.2,

$$R_b = e(e + 1)J + e(e^2 + e + 1)I - (e^2 + 1)A_1 - A_2.$$

The isomorphism type of  $N$  is not determined by the parameters, so  $A_1$  and  $A_2$  are not generally known. But some progress can be made on the assumption that  $N$  is a ‘generalized windmill’, comprising  $e^2 + 2e + 2$  cliques of size  $e + 1$  with one vertex in common. (See also [13, Theorem 4.5].) In that case  $A_1$  can be written as a block-circulant matrix with  $e^2 + 2e + 2$  rows and columns of blocks, each block of size  $e \times e$ .

$$A_1 = \text{bcirc}[J - I \quad O \quad O \cdots O].$$

$A_2$  is a matrix of the same form:

$$A_2 = \text{bcirc}[(e - 1)J + I \quad J \quad J \cdots J].$$

It follows that

$$R_b = \text{bcirc}[U \quad V \quad V \cdots V], \quad U = e(e + 1)^2 I, \quad V = (e^2 + e - 1)J.$$

It is clear that in the ‘generalized windmill’ case  $X_1(v)$  consists of  $e^2 + 2e + 2$  cliques of size  $e$ . Furthermore, it follows from results of Brouwer and Haemers [5] that  $X_2(v)$  is also highly structured. These authors show that the  $(e + 1)^4$  vertices of  $X$  can be partitioned into  $e + 1$  subsets of size  $(e + 1)^3$ , in such a way that each subset induces a graph isomorphic to  $(e + 1)^2$  copies of  $K_{e+1}$ .

This implies that  $X_2(v)$  contains  $e(e+2)$  cliques of size  $e+1$  and  $e(e+1)^2$  cliques of size  $e$ . The required  $B_\flat$  can thus be written as a partitioned matrix conforming with this structure:

$$B_\flat = [Y \ Z],$$

where  $Y$  has  $e^2 + 2e + 2$  rows and  $e(e+2)$  columns of blocks of size  $e \times (e+1)$  and  $Z$  has  $e^2 + 2e + 2$  rows and  $e(e+1)^2$  columns of blocks of size  $e \times e$ .

**Example 6.3** When  $e = 1$  the block matrices  $U$  and  $V$  are the singletons  $[4]$  and  $[1]$ , so we require  $B_\flat B_\flat^T = \text{circ}[4 \ 1 \ 1 \ 1 \ 1]$ . Taking

$$Y = \begin{pmatrix} 00 & 00 & 00 \\ 10 & 10 & 10 \\ 10 & 01 & 01 \\ 01 & 10 & 01 \\ 01 & 01 & 10 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

it is easy to check that  $YY^T + ZZ^T = 3I + J$ , which is the required result. Applying the Reconstruction Theorem  $I - A_P = B^T(I - A_Q)^{-1}B$  it turns out that  $A_P$  is the adjacency matrix of the Petersen graph, and we get the Clebsch graph  $\text{SR}(16, 5, 0, 2)$ .  $\square$

The structural result of Brouwer and Haemers holds for all values of  $e$ . However graphs are known only when  $e+1$  is a prime power. The following construction is given by Brouwer and Haemers, based on a more general method of Ivanov and Shpectorov [10]. Consider the field  $\mathbb{F}_{q^2}$  with the automorphism  $x \mapsto \bar{x} = x^q$ . A  $2 \times 2$  matrix  $M = (m_{ij})$  over  $\mathbb{F}_{q^2}$  is *Hermitian* if  $M^T = \bar{M}$ : explicitly this means that  $m_{11}$  and  $m_{22}$  are in the ground field  $\mathbb{F}_q$  and  $m_{21} = \bar{m}_{12}$ . It follows that there are  $q^4$  such matrices, and we take the set  $\mathcal{H}$  of them to be the vertices of a graph. Note that  $\mathcal{H}$  is a group with respect to addition, but not with respect to multiplication.

Let  $\mathcal{S}$  be the subset of matrices with rank 1, that is

$$\mathcal{S} = \{M \in \mathcal{H} \mid M \neq O \text{ and } \det M = 0\}.$$

If  $M \in \mathcal{S}$  then there  $q-1$  non-zero possibilities for  $m_{11}$  and for each of them there are  $q^2$  possibilities for  $m_{12}$ . The values of  $m_{21}$  and  $m_{22}$  are then determined by the equations  $m_{21} = \bar{m}_{12}$  and  $m_{22} = m_{11}^{-1}m_{12}m_{21}$ . If  $m_{11} = 0$  then  $m_{12} = m_{21} = 0$  and there are  $q-1$  possibilities for  $m_{22}$ . It follows that  $|\mathcal{S}| = (q-1)(q^2+1)$ .

If  $M \in \mathcal{S}$  then  $\alpha M \in \mathcal{S}$  for all  $\alpha \neq 0$ . In particular,  $\mathcal{S} = -\mathcal{S}$ , so we can define the Cayley graph  $(\mathcal{H}, \mathcal{S})$ , in which  $M$  and  $M'$  are adjacent whenever

$M - M' \in \mathcal{S}$ . The first subconstituent  $X_1(O)$  comprises the  $(q - 1)(q^2 + 1)$  members of  $\mathcal{S}$ . They form  $q^2 + 1$  cliques of size  $q - 1$ , each of the form  $\{\alpha M \mid \alpha \neq 0\}$ .

It can be checked that  $c = q(q - 1)$  and so we have a strongly regular graph with parameters

$$n = q^4, \quad k = (q - 1)(q^2 + 1), \quad a = q - 2, \quad c = q(q - 1).$$

The eigenvalue  $e$  is indeed  $q - 1$ , so the parameters  $(a, c, e)$  are as postulated above. It has been shown by various means that the graph is unique in the case  $q = 3$  [5, 13, 14].

Other constructions of the same family are known. For example, the *affine polar graphs* are related to two-weight codes and certain geometrical configurations [6]. Here the vertices of the graph are represented by the elements  $x \in \mathbb{F}_{q^4}$ , and  $x$  and  $y$  are adjacent when  $Q(x - y)$  is a non-zero square, for a suitable quadratic form  $Q$ . Variants of this construction have been studied [3], leading to other graphs with algebraically feasible parameters.

As far as I am aware, these constructions work only when  $e$  is of the form  $q - 1$  with  $q$  a prime power. The fact that the Brouwer-Haemers structure theorem holds for all values of  $e$  raises an obvious question.

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